

# A Projective Interpretation of Some Doubly Special Relativity Theories

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A class of projective actions of the orthogonal group on the projective space is being studied. It is shown that the Fock–Lorentz, and Magueijo–Smolin transformations known as Doubly Special Relativity are such transformations. The formalism easily lead to new type transformations.

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## I. INTRODUCTION

The Lorentz symmetry is one of the cornerstones of modern physics. It is the space-time symmetry of the Standard Model of Particle Physics; and it is the symmetry of space-time as seen by a freely falling observer, in a sufficiently small lab, in any gravitational field. In other words, it is respected both in our quantum theories, and in our classical theories describing gravity. On the other hand, there are some arguments indicating that at high enough energy scales, perhaps the Planck energy scale, Lorentz symmetry might be violated somehow. For example, in some approaches to quantum gravity the spacetime has a polymer-like structure [1], in some the spacetime is non-commutative [2], and in some the spacetime has extra dimensions [3], though there are experimental and observational limits on such violations [4].

To think about this problem, several approaches have been developed. Some physicist have tried to consider the effect of various Lorentz violating terms in the Lagrangian of Particle Physics (see for example [5, 6]). Some have tried to replace a quantum deformation of the Poincaré group, a very important example of which is the  $\kappa$ -Poincaré group [7]. And, some physicists have tried to find a generalization of the Special Relativity.

One of the first generalizations of the Lorentz transformations was introduced several decades ago by V. A. Fock [8], whose motivation was to investigate the implications of the relativity principle—equivalence of inertial frames—as far as possible, that is, relaxing the constancy of the speed of light. Later, S. N. Manida pointed out that these transformations can be interpreted to exhibit a time-varying speed of light [9]. Fock–Lorentz (FL) transformations are transformations of the spacetime. If one uses similar transformations for the energy-momentum space, one obtains the so called Magueijo–Smolin (MS) transformations [10].

In Doubly Special Relativity (abbreviated DSR), the idea is to find transformations which leave an energy (or length) scale invariant [10–15]. Two famous examples of the DSR theories are the Amelino–Camelia [12, 16], and the Magueijo–Smolin (MS) DSRs [10]. It is known that these DSRs are related to  $\kappa$ -Poincaré (see [15]). It is also possible to find more DSR theories from  $\kappa$ -Poincaré formalism [15, 17].

There has been some activities to understand the nature of the nonlinear transformations of DSR theories. To gain insight into these theories, mathematical structures such as non-commutative geometry [14], conformal groups [18, 19], Finsler geometry [20], five dimensional mechanics [21], and perhaps other structures are being invoked.

Few years ago, we argued that some DSRs, namely the FL and the MS DSRs, are merely re-descriptions of Einstein’s Special Theory of Relativity [22, 23], a view which some physicists do not agree (see for example [24]).

Here we want to present a very simple geometrical interpretation of the FL and the MS transformations, which will shed light onto these DSR theories. This geometrical interpretation is so simple that one wonders why it has not been emphasized in the literature [25]. In spite of its simplicity, it enables one to find some new transformations (see eqs. 45, 46, 49, and 50). In this article we are restricting ourselves to introduce this *projective similarity* picture, postponing the study of its implications to a separate article.

It should be emphasized that we are not dealing with  $\kappa$ -deformed DSRs, which are deformations of the Hopf algebra of the generators.

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## II. THE REAL PROJECTIVE SPACE

We have to begin with a short review of the structure of the Real Projective Space. The subject is well known, and there are several textbooks available (see for example [26–28]). But a general knowledge of the basic definitions and properties is enough to follow the arguments.

The Real Projective Space of dimension  $n$ , denoted by  $\mathbb{R}P^n$ , is the space of *rays* in  $\mathbb{R}^{n+1}$ . Any ray in  $\mathbb{R}^{n+1}$  is fully characterized by a pair of antipodal points on the sphere  $S_n = \left\{ x \in \mathbb{R}^{n+1} \mid \|x\| := \sqrt{\sum_1^{n+1} x_i^2} = 1 \right\}$ . Therefore,  $\mathbb{R}P^n$  could be imagined as  $S_n$  divided by the following equivalence relation.

$$x, y \in S_n, \quad x \sim y \iff x = -y \vee x = y. \quad (1)$$

Usually this is written as  $\mathbb{R}P^n = S_n/Z_2$ , where  $Z_2$  is the group consisting of numbers  $\{-1, 1\}$ . The sphere  $S_n$  is naturally endowed with a Riemannian metric, and its curvature is  $+1$ . The above quotient map will induce a metric on  $\mathbb{R}P^n$ . So naturally  $\mathbb{R}P^n$  is a constant positive curvature space.

$\mathbb{R}P^n$ , could also be imagined as  $\mathbb{R}^n$  plus a “(hyper-) plane at infinity” thus: Divide the set of all lines of  $\mathbb{R}^n$  by the equivalence relation of parallelism. The resulting set would have one “point” for each class of parallel lines, which is called the “point at infinity” of that class of parallel lines. The totality of these points form a (hyper)plane, called the “plane at infinity”. Add this plane to the original  $\mathbb{R}^n$  and the result would be  $\mathbb{R}P^n$ . Now any class of parallel lines are “meeting” at a point at infinity, on the “plane at infinity”. We denote this plane at infinity with  $L_\infty$ . When thinking of  $\mathbb{R}P^n$  in this way, one usually forgets that it is a curved space.

## III. ACTION OF $SO(n)$ ON $\mathbb{R}P^n$

The group  $SO(n+1)$  acts naturally on  $S_n$ , and therefore on  $\mathbb{R}P^n$ . In the following we will see that the group  $SO(n)$  also acts on  $\mathbb{R}P^n$ .

The group  $SO(n)$  acts naturally on the space  $\mathbb{R}^n$ . Let  $R$  denotes one such rotation. The action of  $R$  on  $\mathbb{R}^n$  could be extended to a map of  $\mathbb{R}P^n$  onto itself. To this end, we first remind that to each *direction* in the  $\mathbb{R}^n$ , there corresponds *one* point at infinity. Because of this, the result of any rotation by  $\theta = \pi$  would map any point  $P \in L_\infty$  into itself. So that the action of the group  $SO(n)$  on  $L_\infty$  would cover this line *twice*, not once. We say that  $L_\infty$  is mapped onto itself under  $SO(n)$ , which means that it does not mix with any other plane. In  $\mathbb{R}P^n$ , and hence in  $\mathbb{R}^n$ , there are no other planes which are being mapped to themselves under the action of *all* members of  $SO(n)$ .

Now we are going to study another action of  $SO(n)$  on  $\mathbb{R}P^n$  which maps another plane, not necessarily the plane at infinity, onto itself. For simplicity of argument, we first present the case of  $n = 2$ .

In  $\mathbb{R}^2$ , consider a line, not passing through the origin. We have excluded the origin which is the fixed point of  $\mathbb{R}^2$  under  $SO(2)$ .

The most general line, in  $(x, y)$  plane, not passing through the origin has the equation

$$1 + ax + by = 0, \quad (a, b) \neq (0, 0). \quad (2)$$

Let  $\delta$  denotes this line, and let  $H_\delta$  denotes the plane  $\mathbb{R}^2$  minus this line, that is

$$H_\delta = \{(x, y) \in \mathbb{R}^2 \mid 1 + ax + by \neq 0\}. \quad (3)$$

Now consider the following transformation, from  $(x, y)$  plane into  $(X, Y)$  plane:

$$(x, y) \xrightarrow{f} \left( X = \frac{x}{1 + ax + by}, Y = \frac{y}{1 + ax + by} \right). \quad (4)$$

Obviously, this transformation is not defined on  $\delta$ . It can be easily shown that the inverse of this transformation is

$$(X, Y) \xrightarrow{f^{-1}} \left( x = \frac{X}{1 - aX - bY}, y = \frac{Y}{1 - aX - bY} \right), \quad (5)$$

which is not defined on the line  $\Delta$

$$\Delta = \{(X, Y) \mid 1 - aX - bY = 0\}, \quad (6)$$

it is defined on

$$H_\Delta = \{(X, Y) \in \mathbb{R}^2 \mid 1 - aX - bY \neq 0\}. \quad (7)$$

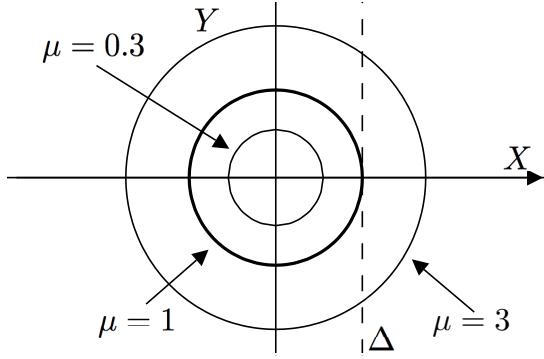


FIG. 1. The orbits, under the action of  $SO(2)$ , when  $SO(2)$  is acting on  $\mathbb{R}P^2$  ordinarily, that is, when it is  $L_\infty$  which is mapped into itself. The line  $\Delta$ , with equation  $X = 1$  is being mapped into  $L_\infty$  by the projective map  $(X, Y) \mapsto \left(\frac{X}{1-X}, \frac{Y}{1-X}\right)$ . The touching circle is the circle with  $\mu = 1$ . Note that circles with  $\mu > 1$  cross  $\Delta$  at two points.

Note that

$$1 - aX - bY = \frac{1}{1 + ax + by}. \quad (8)$$

One can therefore interpret these mappings from  $\mathbb{R}P^2$  to itself, thus:

$$f(\delta) = L_\infty, \quad f(L_\infty) = \Delta, \quad (9)$$

$$f^{-1}(L_\infty) = \delta, \quad f^{-1}(\Delta) = L_\infty. \quad (10)$$

Now, let  $R$  be a rotation acting on the  $(X, Y)$  plane,

$$R : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X \cos \theta - Y \sin \theta \\ X \sin \theta + Y \cos \theta \end{pmatrix} \quad (11)$$

and construct the following mapping of  $\mathbb{R}P^2$  onto itself.

$$S = f^{-1} \circ R \circ f. \quad (12)$$

By straightforward calculation, one can easily show that the map  $S : (x, y) \mapsto (x', y')$  is the following.

$$x' = \frac{x \cos \theta - y \sin \theta}{D} \quad (13)$$

$$y' = \frac{x \sin \theta + y \cos \theta}{D}, \quad (14)$$

where

$$D = 1 + x(a - a \cos \theta - b \sin \theta) + y(b - b \cos \theta + a \sin \theta). \quad (15)$$

To have a clear picture in mind, let's take  $a = 1$  and  $b = 0$  [29]. The denominator of the above transformations now read  $1 + x - x \cos \theta + y \sin \theta$ , which for  $x = -1$  yield  $\cos \theta + y \sin \theta$ , from which it follows that

$$(x = -1, y) \xrightarrow{S_\theta} \left( x' = -1, y' = \frac{y - \tan \theta}{1 + y \tan \theta} \right), \quad (16)$$

which explicitly shows that under the above transformation the line  $\delta$  with equation  $1 + x = 0$ , is mapped into itself. This line is not “invariant”, in the sense that its points are not fixed. Also note that  $\theta$  and  $\theta + \pi$  map  $y$  to the same point  $y'$ , and therefore, for  $\theta \in [-\pi, \pi]$  the line  $x = -1$  is mapped to itself twice.

The line  $\Delta$  is the line  $X = 1$ . In the  $(X, Y)$  plane, the orbits, under the action of  $SO(2)$  are the circles with center at the origin, which are characterized by a real non-negative number  $\mu$ .

$$C_\mu := \{(X, Y) \mid X^2 + Y^2 = \mu\}. \quad (17)$$

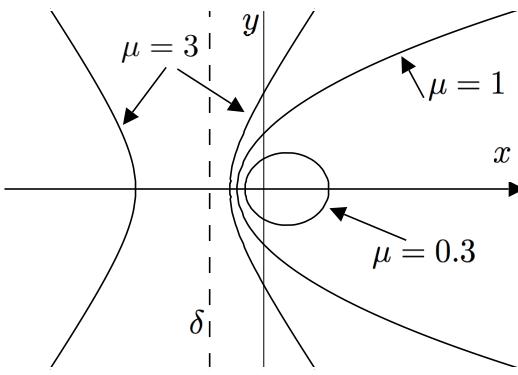


FIG. 2. The orbits, under the action of  $SO(2)$ , when  $SO(2)$  is acting on  $\mathbb{R}P^2$  such that the line  $\delta$ , with equation  $x = -1$  is mapped into itself. The image of the touching circle, is now the parabola  $\mu = 1$ . The line  $\delta$  is invariant under this action. As is seen from the two-sheet hyperbola, for some values of the rotation parameter  $\theta$ , points on one side of  $\delta$  will move to the other side. Also note that when  $\mu \rightarrow \infty$ , the two sheets of the hyperbola will approach  $\delta$ , and will cover it twice.

Among these circles, there is one, and only one, which is tangent to the line  $\Delta$ . We call it the *tangential circle*. It is a straightforward calculation of elementary analytic geometry to show that for tangential circle

$$\mu = \frac{1}{a^2 + b^2}. \quad (18)$$

The corresponding sets, invariant under the action of  $SO(2)$  on the  $\mathbb{R}P^2$ , are the following conics.

$$\Gamma_\mu := \left\{ (x, y) \mid \frac{x^2 + y^2}{(1+x)^2} = \mu \right\}. \quad (19)$$

Some remarks are worthy of mention. First, though the line  $\delta$  cuts  $\mathbb{R}^2$  into two disjoint halves, the action of  $SO(2)$  just constructed does not leave either of these halves invariant. This could be seen in Fig. 2. Second, though the group  $SO(2)$  is compact, the orbits  $\Gamma_\mu$  for  $\mu > 1$  are hyperbolas, which, are not compact. The reason is that the map  $f$  is singular on  $\delta$ .

#### IV. THE GENERAL CASE

To deal with  $\mathbb{R}P^n$ , for  $n > 2$ , and the more general case of  $SO(1, n-1)$ , let's use matrix notation. We define

$$a = \begin{bmatrix} a^0 \\ a^1 \\ \vdots \\ a^{n-1} \end{bmatrix}, \quad x = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^{n-1} \end{bmatrix}. \quad (20)$$

and introduce the metric

$$H = [\eta_{\mu\nu}] = \text{diag}(-1, 1, \dots, 1). \quad (21)$$

The transformations  $f$  and  $f^{-1}$  have now a compact form—

$$f : x \mapsto X = \frac{x}{1 + a^T H x}, \quad (22)$$

$$f^{-1} : X \mapsto x = \frac{X}{1 - a^T H X}, \quad (23)$$

where a superscript  $T$  means matrix transposition. The above formula for  $f^{-1}$  could be easily proved by substitution.

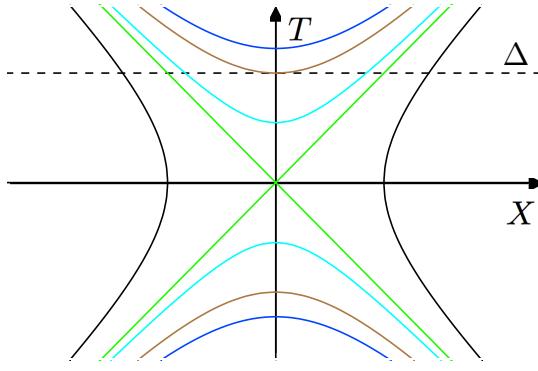


FIG. 3. Orbits of  $SO(1,1)$  in  $X^\mu$  space, for  $a^\mu = (-1, 0)$ . The dashed line with equation  $T = 1$  is the line  $\Delta$ . The touching hyperbola is the one tangent to the dashed line. Note that all the orbits are asymptotic to the null directions.

Now let's calculate the effect of  $S = f^{-1} \circ \Lambda \circ f$  on  $x$ , where  $\Lambda \in SO(1, n-1)$ . Under  $f$  we have  $X = \frac{x}{1+a^T H x}$ , and under  $R$  we have  $X' = \frac{\Lambda x}{1+a^T H x}$ . Finally, under  $f^{-1}$  we have  $x' = \frac{X'}{1-a^T H X'}$ . So we need to calculate  $1 - a^T H X'$  which is straightforward.

$$1 - a^T H X' = 1 - a^T H \left( \frac{\Lambda x}{1 + a^T H x} \right) \quad (24)$$

$$= 1 - \frac{a^T H \Lambda x}{1 + a^T H x} \quad (25)$$

$$= \frac{1 + a^T H x - a^T H \Lambda x}{1 + a^T H x}. \quad (26)$$

Therefore, we have

$$\frac{1}{1 - a^T H X'} = \frac{1 + a^T H x}{1 + a^T H (\mathbb{I} - \Lambda) x}, \quad (27)$$

where  $\mathbb{I}$  is the unit  $n \times n$  matrix. Now, for  $S$  we have

$$x' = \frac{X'}{1 - a^T H X'} \quad (28)$$

$$= \frac{\Lambda H x}{1 + a^T H x} \cdot \frac{1 + a^T H x}{1 + a^T H (\mathbb{I} - \Lambda) x} \quad (29)$$

$$= \frac{\Lambda x}{1 + a^T H (\mathbb{I} - \Lambda) x}, \quad (30)$$

which, in the usual notation familiar in physics—with the Einstein summation convention understood—would read

$$x'^\mu = \frac{\Lambda_\nu^\mu x^\nu}{1 + a_\alpha (\delta_\beta^\alpha - \Lambda_\beta^\alpha) x^\beta}. \quad (31)$$

Perhaps the skeptical reader should check directly that the result of combining two such maps, first with  $\Lambda$ , and followed with  $\bar{\Lambda}$  would be one with  $\bar{\Lambda} \Lambda$ . This is obvious from the construction of  $S = f^{-1} \circ \Lambda \circ f$ , but can also be checked by straightforward calculation.

The generators of the transformation (31) are

$$\tilde{M}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + (a_\mu x_\nu - a_\nu x_\mu) x^\sigma \partial_\sigma. \quad (32)$$

These operators satisfy the Lie algebra  $so(1, n)$ , because the acting group is  $SO(1, n)$ .

## V. THE CASE $SO(1,1)$ ACTING ON $\mathbb{R}P^2$

Under the action of  $SO(1,1)$  the form  $-T^2 + X^2$  is invariant. Now remind that

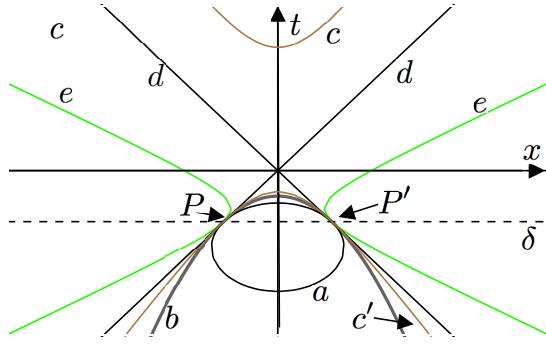


FIG. 4. Orbits of  $SO(1,1)$  on  $\mathbb{R}P^2$ , for  $a^\mu = (-1, 0)$ . The invariant plane is  $\delta$  (the set  $t = -1$ ). For  $\mu < -1$  the orbit is an ellipse (labeled  $a$ ). For  $\mu = -1$  the orbit is a parabola ( $b$ ). For  $-1 < \mu < 0$  the orbit is a spacelike hyperbola of two sheets ( $c$  and  $c'$ ). For  $\mu = 0$  the orbit is a cone ( $d$ ). For  $\mu > 0$  the orbit is a time-like hyperbola of two sheets ( $e$ ). Note that all these orbits intersect at the points  $P$  and  $P'$ , which are the images (under  $f^{-1}$ ) of the null directions in the  $X^\mu$  space, and remember that all the orbits of  $SO(1,1)$  in  $X^\mu$  space, are asymptotic to the null directions.

- $-T^2 + X^2 = \rho < 0$  is a two-sheet hyperbola.
- $-T^2 + X^2 = 0$  is the light cone, consisting of two straight lines passing through the origin.
- $-T^2 + X^2 = \rho > 0$  is a two-sheet hyperbola.

First we must specify the vector  $a$ , which is either timelike ( $a_\mu a^\mu < 0$ ), or lightlike ( $a_\mu a^\mu = 0$ ), or spacelike ( $a_\mu a^\mu > 0$ ).

### A. A timelike $a$

A simple form of such an  $a^\mu$  is  $a^\mu = (-1, 0)$ . The mappings  $f$  and  $f^{-1}$  now read

$$f : x^\mu \mapsto X^\mu = \frac{x^\mu}{1+t} \quad (33)$$

$$f^{-1} : X^\mu \mapsto x^\mu = \frac{X^\mu}{1-T} \quad (34)$$

The planes  $\delta$  and  $\Delta$  are the planes

$$\delta : \quad 1 + a_\mu x^\mu = 1 + t = 0, \quad (35)$$

$$\Delta : \quad 1 - a_\mu X^\mu = 1 - T = 0. \quad (36)$$

The touching hyperbola (in  $X^\mu$  space) would be the sheet  $T > 0$  of the hyperboloid  $-T^2 + X^2 = -1$ , which by substitution  $X^\mu = x^\mu/(1+t)$ , is mapped to the conic  $-t^2 + x^2 = -(1+t)^2$ , which is the parabola  $t = -\frac{1}{2}(x^2 + 1)$ .

More generally, consider the conic  $-t^2 + x^2 = \mu(1+t)^2$ . For  $\mu = -1$  this is the parabola mentioned above. For  $\mu \neq -1$  one can write it in the canonical form

$$\frac{x^2}{1+\mu} - \left(t + \frac{\mu}{1+\mu}\right)^2 = \frac{\mu}{(1+\mu)^2}. \quad (37)$$

We therefore have

- for  $-\infty < \mu < -1$ , it is an ellipse. As a submanifold of  $\mathbb{R}^2$  with the ordinary Euclidean topology, this set is compact. This seems strange, since the group  $SO(1,1)$  is not compact, and we are saying that this ellipse is the image of  $SO(1,1)$ . Here we should remember that the map  $f$  is singular.
- for  $\mu = -1$ , it is a parabola.
- for  $-1 < \mu < 0$ , it is a space-like hyperbola, consisting of two sheets.
- for  $\mu = 0$ , it is a 1 dimensional cone, consisting of two lines.

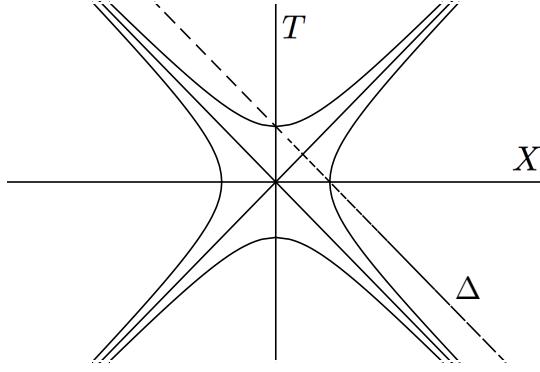


FIG. 5. The case  $a^\mu = (-1, 1)$ .

- for  $0 < \mu < \infty$ , it is a timelike hyperbola, consisting of two sheets.

Now let's look at the transformation (31) for the specific case of a boost in the  $x^1$  direction, in which we have

$$[\Lambda_\nu^\mu] = \begin{bmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{bmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}. \quad (38)$$

For  $a^0 = -1$ , and  $a^i = 0$ , we have  $a_0 = 1$ , and  $a_i = 0$ , and it follows that

$$1 + a_\alpha (\delta_\beta^\alpha - \Lambda_\beta^\alpha) x^\beta = 1 - t(\gamma - 1) + \gamma v x, \quad (39)$$

and therefore the transformation  $S = f^{-1} \circ \Lambda \circ f$  reads

$$t' = \frac{\gamma(t - vx)}{1 - t(\gamma - 1) + \gamma vx}, \quad (40)$$

$$x' = \frac{\gamma(x - vt)}{1 - t(\gamma - 1) + \gamma vx}. \quad (41)$$

This is the Fock-Lorentz transformation. Using the same transformation on the energy-momentum space, one could easily get the Magueijo-Smolin transformation, the motivation of which was to have transformations with an invariant energy scale [10].

### B. A spacelike $a$

A simple form of such an  $a^\mu$  is  $a^\mu = (0, 1)$ . The mappings  $f$  and  $f^{-1}$  now read

$$X^\mu = \frac{x^\mu}{1+x}, \quad x^\mu = \frac{X^\mu}{1-X}. \quad (42)$$

The planes  $\delta$  and  $\Delta$  are

$$\delta : 1+x=0, \quad \Delta : 1-X=0. \quad (43)$$

The touching hyperbola (in  $(T, X)$  plane) would be the sheet  $X > 0$  of the hyperbola  $-T^2 + X^2 = 1$ . The image of this hyperbola, under the map  $f^{-1}$  is the parabola  $x = -\frac{1}{2}(t^2 + 1)$ . It is obvious that all the other conics also can be derived from the previous case of a timelike  $a$ , by just a  $\frac{\pi}{2}$  rotation (Euclidean rotation) in the  $(t, x)$  plane. Now the ellipse of fig 4 will be a closed curve, turning back in time. A physical interpretation of such a curve is very difficult.

Anyhow, let's find the “projective” transformation (31). We now have

$$1 + a_\alpha (\delta_\beta^\alpha - \Lambda_\beta^\alpha) x^\beta = 1 - x(\gamma - 1) + \gamma v t, \quad (44)$$

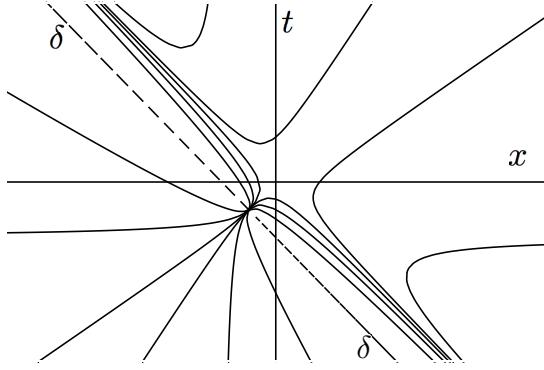


FIG. 6. The orbits of  $SO(1, 1)$  under the projective action for  $a^\mu = (-1, 1)$ .

from which it follows that the boost in the  $x$  direction, read

$$t' = \frac{\gamma(t - vx)}{1 - x(\gamma - 1) + \gamma vt}, \quad (45)$$

$$x' = \frac{\gamma(x - vt)}{1 - x(\gamma - 1) + \gamma vt}. \quad (46)$$

As far as we know, this transformation is new.

### C. A lightlike $a$

Consider  $a^\mu = (-1, 1)$  which leads to  $1 + a \cdot x = 1 + t + x$ , and  $1 - a \cdot X = 1 - T - X$ . Now the line  $\delta$  is  $1 + t + x = 0$ , and the line  $\Delta$  is  $1 - T - X = 0$ . There is no touching hyperbola, since none of the hyperbolas are now tangent to  $\Delta$ . The orbits of  $SO(1, 1)$  under this projective action are

$$t^2(\mu + 1) + x^2(\mu - 1) + 2\mu(t + x + tx) + \mu = 0 \quad (47)$$

which are being drawn in fig 6. We are not going to describe these conics. Let's only write the projective transformation for this case.

$$1 + a_\alpha (\delta_\beta^\alpha - \Lambda_\beta^\alpha) x^\beta = 1 + (t + x)(1 - \gamma + \gamma v), \quad (48)$$

from which it follows that

$$t' = \frac{\gamma(t - vx)}{1 + (t + x)(1 - \gamma + \gamma v)}, \quad (49)$$

$$x' = \frac{\gamma(x - vt)}{1 + (t + x)(1 - \gamma + \gamma v)}. \quad (50)$$

As far as we know, this transformation is also new.

## VI. ABOUT THE METRIC

Using the map

$$x \mapsto X = \frac{x}{1 + a \cdot x} \quad X^\mu = \frac{x^\mu}{1 + a_\alpha x^\alpha} \quad (51)$$

one can pull back the flat metric  $ds^2 = dX \cdot dX = \eta_{\mu\nu} dX^\mu dX^\nu$  to the space  $H_\delta$ . The result would be

$$ds^2 = \frac{dx \cdot dx}{(1 + a \cdot x)^2} - \frac{2a \cdot dx \cdot x \cdot dx}{(1 + a \cdot x)^3} + \frac{x \cdot x \cdot (a \cdot dx)^2}{(1 + a \cdot x)^4}. \quad (52)$$

This metric, being the pullback of the flat metric  $\eta_{\mu\nu}$ , has zero Gaussian curvature. However, it must be noted that because of the Gauss-Bonnet theorem,  $\mathbb{R}P^2$  does not admit a global Riemannian metric with zero Gaussian curvature—remind that the Euler characteristic of  $\mathbb{R}P^2$  is 1. Therefore, for  $\mathbb{R}P^2$ , the pullback of the Euclidean metric could not be considered as a global metric on  $\mathbb{R}P^2$ —it is not well defined on  $\delta$ —and the singularity is not removable by change of coordinates. It seems to us that the case of the Minkowski metric on  $\mathbb{R}^4$ , being pulled back to the corresponding  $H_\delta$  has also this pathology; but at the moment we have no rigorous proof of that.

## VII. GENERALIZING

Generalizing to any matrix group acting on  $\mathbb{R}^n$ , for example to  $SO(m, n - m)$ , and in particular to  $SO(1, n - 1)$ , is straightforward. In the case of  $SO(1, n - 1)$ , three types of transformations will be obtained, depending on the sign of  $a \cdot a$ . The case of timelike  $a^\mu = (-1, 0, 0, 0)$  will lead to the FL and MS transformations. The only nontrivial part of the generalization is the topology of the orbits, which we are not going through in this paper.

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